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FAST TRACK COMMUNICATION

Eigenproblems of large powers of the Laplacian in bounded domains

A Ramani[1], **B Grammaticos**[2] **and Y Pomeau**[3]

[1] CPT, Ecole Polytechnique, CNRS, UMR 7644, 91128 Palaiseau, France
[2] IMNC, Université Paris VII-Paris XI, CNRS, UMR 8165, Bât. 104, 91406 Orsay, France
[3] Laboratoire de Physique Statistique de l'Ecole normale supérieure, 24 Rue Lhomond, 75231 Paris Cedex 05, France

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Abstract
We present a method for computing the spectrum of large powers of the Laplacian in a bounded domain restricting ourselves to the one- and three-dimensional cases. Since it does not seem possible to obtain information on the eigenvalues directly from the transcendental equation that gives the spectrum, we introduce a Wallis-inspired method. We obtain the expansion of the eigenfunction and the eigenvalues in power series where the inverse of the power at which the Laplacian is raised plays the role of the small parameter. We compare these eigenvalues to those obtained through a simple variational approach and remark that the latter offers an excellent approximation to the exact result.

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The eigenvalues and eigenfunctions of real symmetric operators are rarely given by explicit expressions. In most cases one has to rely on various approximation schemes and asymptotic methods in order to obtain quantitative results. Among the many asymptotics that have been investigated (predominantly the limit of highly excited modes, to use the terminology of quantum mechanics as elsewhere in this communication), the one of a very large power of the Laplacian, if one considers the problem of eigenvalues of differential operators in bounded domains, seems to have been overlooked.

The present communication addresses this question in a simple setting. We begin by considering the one-dimensional eigenvalue problem of the Nth power of the Laplacian operator:

$$\frac{d^{2N}}{dx^{2N}}u = (-1)^N \lambda^{2N} u. \tag{1}$$

The boundary conditions to be satisfied by the eigenfunction u are $u^{(j)}(\pm 1) = 0$ for $j = 0, \ldots, (N-1)$ and $u^{(j)}$ stands for the jth derivative of u. Given the invariance of (1) and its boundary conditions under the symmetry $x \to -x$, the eigenfunctions are characterized by a definite parity i.e., $u(-x) = \pm u(x)$. In order to simplify the approach we shall consider separately the spectrum of (1) for the even and odd eigenfunctions.

The case $N = 1$ is elementary. For the even part of the spectrum the (unnormalized) eigenfunction is just $u = \cos \lambda x$. The boundary condition is $\cos \lambda = 0$, leading to the spectrum $\lambda_k = \frac{\pi}{2} + k\pi$ where λ_0 is the ground state with k numbering the excited states *within the even parity spectrum*.

For $N = 2$, equation (1) reads $u'''' = \lambda^4 u$, leading to an (even) eigenfunction of the form $u = \cos \lambda x + a \cosh \lambda x$. Implementing the boundary conditions $u(\pm 1) = u'(\pm 1) = 0$ we find $a = -\cos \lambda / \cosh \lambda = \sin \lambda / \sinh \lambda$ leading to the equation for the spectrum $\tan \lambda + \tanh \lambda = 0$. While the whole spectrum cannot be obtained analytically, its asymptotic form for large k is $\lambda_k = \frac{3\pi}{4} + k\pi$ since $\tanh \lambda_k$ will rapidly approach unity.

For higher, but moderate values of N, we can obtain, with the help of computer algebra, the equation for the spectrum, which involves trigonometric and hyperbolic functions. We have indeed performed this study for values of N up to 10 and obtained numerically the eigenvalues. They converge very rapidly when k increases to their asymptotic value $\lambda_k = \frac{(N+1)\pi}{4} + k\pi$. In particular for $N = 10$ we have obtained the following values for $4\lambda/\pi$: 10.748, 14.966, 19.995, 22.999, ... The convergence to the asymptotic values $N + 1 + 4k$ is exponential (and, in fact, even the ground state is not very far from the estimate 11). An even more spectacular convergence is obtained for $N = 2$, where we found for $4\lambda/\pi$: $3 + 1 \times 10^{-2}$, $7 + 2 \times 10^{-5}$, $11 + 4 \times 10^{-8}$, $15 + 7 \times 10^{-11}$, again an exponential convergence. This can be understood from the structure of the exact eigenfunction by isolating the dominant contribution. It turns out that in all cases the ratio of two consecutive departures of the calculated $4\lambda/\pi$ from the predicted integers decreases by a factor of order of magnitude $\exp\left(2\pi \sin \frac{\pi}{N}\right)$.

What is particularly interesting is that the one-dimensional approach can be easily extended to the study of the higher powers of the Laplacian in a three-dimensional spherical domain. Starting from the spherical Laplacian

$$\Delta w = \frac{1}{r}\frac{\mathrm{d}^2}{\mathrm{d}r^2}(rw) - \frac{L(L+1)}{r^2} w \tag{2}$$

we see that if we define the function $u = rw$, one can write the eigenfunction of the Nth power of the Laplacian in terms of u as

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{L(L+1)}{r^2}\right)^N u = (-1)^N \lambda^{2N} u. \tag{3}$$

The set of boundary conditions $u^{(j)}(1) = 0$ for $j = 0 \cdots (N-1)$ (where $u^{(j)}$ stands for the jth derivative of u with respect to r) is equivalent to the set $w^{(j)}(1) = 0$ for $j = 0 \cdots (N-1)$. We remark readily that the centrifugal term is absent when $L = 0$ (and also when $L = -1$). In the $L = 0$ case the problem coincides, on the half-positive axis, with the one-dimensional one. However, one notes that while the $L = 0$ eigenfunction w does not vanish at the origin, the auxiliary variable $u = rw$ does, with a nonzero derivative. With this boundary condition, this corresponds to an *odd* solution of the one-dimensional problem. For instance, in the $N = 1$ case, the eigenfunctions of the Laplacian for $L = 0$ are $w = \frac{\sin \lambda r}{r}$, corresponding to $u = \sin \lambda r$, which is the odd eigenfunctions in the one-dimensional case. So we can lump the treatment of the latter case together with the spherical case $L = 0$. It turned out that we can present a unified description of our results for the *even* eigenfunctions in the one-dimensional case together with the spherical case by formally giving to L the value -1. The odd one-dimensional case is identical to the $L = 0$ spherical case, as explained above.

The problem we will consider is therefore $\Delta^N u = (-1)^N \lambda^{2N} u$ where the Laplacian is now defined as $\Delta = \frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{L(L+1)}{x^2}$, $L = -1$ corresponding to the even one-dimensional case.

Although this cannot be completely excluded, there seems to be little hope to get any information on the spectrum by looking at the behaviour of the large-N limit of the transcendental equation that yields directly the eigenvalues. So we shall instead consider the problem by a Wallis-inspired [1] method, namely by looking at the Taylor expansion of the eigenfunctions near $x = 0$ and by writing the eigenproblem as a condition for the coefficients of this expansion.

We start by introducing a variational problem which allows us to obtain a good approximation to the ground-state eigenvalue. We consider the functional $I = \int_{-1}^{1} (\Delta^N U - EU)^2 \,\mathrm{d}x$ for the test function U in the one-dimensional case, and $I = 4\pi \int_0^1 (\Delta^N U - EU)^2 \,\mathrm{d}r$ in the spherical case, where the r^2 term in the volume element has been absorbed into the change of variables to the auxiliary $U = rW$. Given the symmetry property of the eigenfunctions (either odd or even), the one-dimensional case can be rewritten as $I = 2\int_0^1 (\Delta^N U - EU)^2 \,\mathrm{d}x$. Because a fixed prefactor plays no role in a variational problem, both cases can be unified into the single form $I = \int_0^1 (\Delta^N U - EU)^2 \,\mathrm{d}x$ where it is understood that x stands for r in the spherical case. Minimizing the functional with respect to the eigenvalue E we obtain

$$E = \frac{\int_0^1 U(\Delta^N U)\,\mathrm{d}x}{\int_0^1 U^2\,\mathrm{d}x}. \tag{4}$$

Next we introduce the appropriate test function. For the one-dimensional even case, we choose $U = (1 - x^2)^N$. We remark readily that it satisfies the boundary conditions: it vanishes, together with its $(N - 1)$ first derivatives, at $x = \pm 1$. For the odd case we will use $U = x(1 - x^2)^N$. Note that it is just x (which, we remind the reader, represents r in the spherical case) multiplying an appropriate function $W = (1 - x^2)^N$ for the $L = 0$ spherical case, and as expected this U is just the appropriate auxiliary function in that case. For general values of L, a test function, in terms of W, should be $W = x^L(1 - x^2)^N$, corresponding to $U = x^{L+1}(1 - x^2)^N$. The same expression is thus valid for all cases, *including the one-dimensional even case* represented by $L = -1$. Using this expression the variational estimate $\tilde{E}_0$ for the ground-state energy E_0 can be computed in closed form. We find

$$\tilde{E}_0 = (2N)!\frac{(2N)!!}{(2N-1)!!}\frac{(4N+2L+3)!!}{(4N)!!(2L+1)!!(2N+2L+3)} \tag{5}$$

where we take the usual definition $(-1)!! = 0!! = 1$. It is perhaps more interesting to give an approximate but simpler expression for $\tilde{E}_0$ using the Stirling formula for some of the factorials that appear in (5). We obtain thus

$$\tilde{E}_0 = \sqrt{2}(2N)!\frac{(4N)^{L+1}}{(2L+1)!!}\left(1 + \frac{4L^2-7}{16N} + \frac{\alpha(L)}{512N^2} + \cdots\right) \tag{6}$$

where $\alpha(L) = 16L^4 - \frac{64}{3}L^3 + 72L^2 + \frac{1552}{3}L + 465$.

While the variational eigenfunction is a very good approximation of the true one, it remains that it is still approximate. In order to obtain a more precise estimate of the eigenfunction u, we introduce a corrective factor Φ such that $u_{N,L}(x) = U_{N,L}(x)\Phi_{N,L}(x)$ where Φ is a slowly varying function of x, which will be found to be, at leading order in $1/N$, a function of x^2/N. We thus expand $\Phi_{N,L}(x)$ in a series in powers of $1/N$ involving auxiliary functions $\Phi_{N,L}(x) = \sum_{i=0}^{\infty} N^{-i}\phi_{L,i}(x^2/N)$. Next we expand $\phi_i(x^2/N)$ in a formal power series in x^2/N: $\phi_{L,i}(x^2/N) = \sum_{m=0}^{\infty} a_{L,i,m}\left(\frac{x^2}{N}\right)^m$. The idea behind this two-staged expansion will become clearer in what follows. At a given L and for each value of i, the $a_{L,i,m}$ satisfy an infinite number of linear equations (in general, nonhomogeneous) provided the $a_{L,j,m}$ are known for $j < i$.

For $L = -1$ and $i = 0$ we have (dropping the index $L = -1$)

$$(-1)^N \Delta^N (1-x^2)^N \phi_0(x^2/N) = E_0(1-x^2)^N(\phi_0(x^2/N) + \mathcal{O}(1/N)). \qquad (7)$$

The logic we used is the following. On the left-hand side, we rewrite $(-1)^N(1-x^2)^N$ as $(x^2-1)^N$ and expand it as $\sum_p (-1)^p \binom{N}{p} x^{2(N-p)}$. The summation is only on $(N+1)$ terms but we assume N very large and treat the sum as infinite. Also, we expand each binomial factor as $\binom{p}{N} = \frac{N^p}{p!}\left(1 - \frac{p(p-1)}{2N} + \cdots\right)$. So the left-hand side of equation (7) is, at leading order

$$\Delta^N \left(\sum_p (-1)^p \frac{N^p}{p!} x^{2(N-p)}\right)\left(\sum_{m=0}^{\infty} a_{0,m}\left(\frac{x^2}{N}\right)^m\right).$$

All powers of x less than $2N$ are annihilated by the Nth power of the Laplace operator and can be ignored. The coefficient of $x^{2(N+r)}$ is obtained by choosing, in the second sum, the value $m = p + r$. So the relevant quantity is

$$\Delta^N \sum_r \frac{x^{2(N+r)}}{N^r}\left(\sum_p (-1)^p \frac{a_{0,p+r}}{p!}\right) = \sum_r \frac{(2N+2r)!}{N^r(2r)!} x^{2r}\left(\sum_p (-1)^p \frac{a_{0,p+r}}{p!}\right).$$

At leading order, the $2r$ factors beyond $2N$ in $(2N+2r)!$ can be identified to just $(2N)^{2r}$. (Subdominant terms in $1/N$ will enter, as inhomogeneous terms, into the equations for the $a_{i,m}$, $i > 0$.) On the right-hand side we expand in the same way $(1-x^2)^N$ as $\sum_r (-1)^r \frac{N^r}{r!} x^{2r}$. Here, at leading order, only the first term in ϕ_0, namely $a_{0,0}$, is needed. Comparing the powers of x^{2r} we have to solve

$$(2N)! \frac{2^{2r}}{(2r)!}\left(\sum_p (-1)^p \frac{a_{0,p+r}}{p!}\right) = \frac{(-1)^r}{r!} E_0 a_{0,0}. \qquad (8)$$

We found that all these (homogeneous) equations are satisfied by the quantities $a_{0,m} = (-1)^m \frac{(2m-1)!!}{2^{2m}}$ for the eigenvalue $E_0 = \sqrt{2}(2N)!$ where we normalize with $a_{0,0} = 1$. The case $r = 0$ just means that $\sum_p \frac{(2p-1)!!}{2^{2p} p!} = \sqrt{2}$, and indeed one can recognize the expansion of $(1+y)^{-1/2}$ for $y = -1/2$. It is elementary to see that similar expansions prove that the equations for all r are satisfied too. The series for ϕ_0 is asymptotic, but it can be Borel resummed. From the well-known expression of the error function, we find that $\phi_0 = \sqrt{\pi} z\, \mathrm{e}^{z^2}(1 - \mathrm{erf}(z))$ where $z = \sqrt{2N}/x$. All corrective terms in $1/N^i$ enter, as inhomogeneous terms, into the infinite system of equations for the $a_{i,m}$, $i > 0$.

For higher values of L we find that $a_{L,0,m} = (-1)^m \frac{(2m+2L+1)!!}{2^{2m}}$ (and the equations being linear we do not have to normalize to $a_{L,0,0} = 1$ for $L > 0$). The coefficients $a_{L,0,m}$ for each L turn out to be the building blocks for the construction of the $a_{L,i,m}$ for all values of i (and also, as we will explain below, of the excitation degree k). Indeed, we are able to write the $a_{L,i,m}$ as the product of the $a_{L,0,m}$ multiplied by a polynomial in m of degree $2i$, provided we also expand the eigenvalue in powers of $1/N$ around the zeroth-order value $\sqrt{2}(2N)!(4N)^{L+1}/(2L+1)!!$. The infinitely many equations will be satisfied by the $a_{L,i,m}$ by finding just the (finite number of) coefficients of this polynomial plus the correction to the eigenvalue at that order. For instance we find that $a_{L,1,m} = -a_{L,0,m} m(m - 2L - 6)/4$ (the first correction to the eigenvalue appearing in equation (9)).

In practice the calculations do become cumbersome beyond a certain order. We have thus limited our calculations to fourth order for $L = -1$, and to second order for $L \geqslant 0$, for a sufficiently large number of L's to convince ourselves that we found the correct expansion of the eigenvalue. We found

$$E_0 = \sqrt{2}(2N)!\frac{(4N)^{L+1}}{(2L+1)!!}\left(1 + \frac{4L^2-7}{16N} + \frac{\beta(L)}{512N^2} + \cdots\right) \tag{9}$$

where $\beta(L) = 16L^4 - \frac{64}{3}L^3 - 56L^2 + \frac{400}{3}L + 177$. Thus the variational expression we found above coincides with the exact value up to the first order in $1/N$, the relative difference appearing only at order $1/N^2$, with value, at that order:

$$\frac{\tilde{E}_0}{E_0} - 1 = \frac{\alpha(L) - \beta(L)}{512N^2} = \frac{(2L+3)^2}{16N^2}.$$

For the case $L = -1$ we have computed two more orders and found

$$E_0 = \sqrt{2}(2N)!\left(1 - \frac{3}{16N} + \frac{25}{512N^2} + \frac{375}{8192N^3} - \frac{8197}{524\,288N^4} + \cdots\right). \tag{10}$$

We can give the expression of the first four $a_{i,m}$ for the case $L = -1$ in terms of $a_{0,m}$ given above:

$$\begin{aligned}
a_{1,m} &= -a_{0,m}m(m-4)/4\\
a_{2,m} &= a_{0,m}m(3m^3 - 68m^2 + 36m - 1)/96\\
a_{3,m} &= -a_{0,m}m(m^5 - 56m^4 + 312m^3 + 143m^2 + 218m - 1392)/384\\
a_{4,m} &= a_{0,m}m(15m^7 - 1560m^6 + 27\,320m^5 - 30\,942m^4 - 13\,120m^3 - 59\,880m^2\\
&\qquad - 97\,915m - 31\,638)/92160.
\end{aligned}$$

The case of the excited states can be treated along the same lines. In addition to the slowly varying factor Φ we need an extra corrective factor, namely a function Ψ of Nx^2. The latter is just a polynomial in x^2 of degree exactly k, the coefficients of which must be expanded in powers of $1/N$ (the coefficient of x^{2p} starting as N^p), the number of needed terms depending on the desired value of i. So for a given value of i the total number of unknowns to be computed is always finite: the polynomial giving $a_{i,m}$ is now of degree $2i + 2k$ and we also need the expansions of both Ψ and the eigenvalue. This will be treated in a future publication. Let us just give the preliminary result for the first excited state for $L = -1$ up to first order in $1/N$. The eigenvalue at this order is

$$E_1 = 8N^2\sqrt{2}(2N)!\left(1 - \frac{15}{16N}\right). \tag{11}$$

Here we do not have a variational expression to compare it to. We find $a_{0,m}^{(k=1)} = a_{0,m}^{(k=0)}(2m^2 + 3m + 1)$, $a_{1,m}^{(k=1)} = -a_{0,m}^{(k=0)}(2m^2 + 3m + 1)m(m-8)/4$ and in that case $\Psi = (1 - x^2(4N + 3 + \cdots))$ where the degree in x is just 2 but the expansion in $1/N$ extends to infinity. Only the terms written here are needed to compute $a_{1,m}^{(k=1)}$ and the eigenvalue at the order given above.

In this communication we have considered the problem of finding the eigenvalues and eigenfunctions of a large power of the Laplacian. We have presented its solution in the case of the lowest eigenvalue ('ground state' in the quantum-mechanical terminology). The study of the excited states will be the subject of some future publication. Our presentation was limited here to the analysis of the one- and three-dimensional cases but our method can easily be extended to the case of an arbitrary bounded domain with Dirichlet boundary conditions. We plan to return to this question in some future work. Finally, the convergence of the expansions of the eigenvalues and eigenfunctions has not been addressed in this work and remains an open problem which, we hope, might stimulate the readers of this communication.

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